Exercises Section 6

1. $\rho \to \mathcal{E}(\rho) = \sum_j A_j \rho A_j^\dagger$

$\mathrm{Tr}[\mathcal{E}(\rho)] = 1 \Rightarrow \mathrm{Tr}\left[\sum_j A_j \rho A_j^\dagger\right] = \sum_j \mathrm{Tr}\left[A_j^\dagger A_j \rho\right] = 1$

true for any ρ : $\sum_j A_j^\dagger A_j = \mathbb{I}$.

Let $M \neq M^\dagger$ and $M|\phi_j\rangle = m_j |\phi_j\rangle$ with $m_j \in \mathbb{C}$.

$m_j = a_j + i b_j$ with $a_j, b_j \in \mathbb{R}$

$$M|\phi_j\rangle = m_j|\phi_j\rangle = (a_j + ib_j)|\phi_j\rangle = a_j|\phi_j\rangle + i b_j|\phi_j\rangle$$
$$= A|\phi_j\rangle + iB|\phi_j\rangle = (A+iB)|\phi_j\rangle$$

now, $A = A^\dagger$ and $B = B^\dagger$ because of real eigenvalues.

$L_0 = -\frac{1}{2}\sum_j L_j^\dagger L_j$: $\mathrm{Tr}\left(\frac{d\rho}{dt}\right) = 0$ because $\mathrm{Tr}(\rho) = 1$.

$$\mathrm{Tr}\left(\frac{d\rho}{dt}\right) = \mathrm{Tr}\left[-\frac{i}{\hbar}[H,\rho] + \{L_0,\rho\} + \sum_{j\neq 0} L_j \rho L_j^\dagger\right]$$
$$= \mathrm{Tr}\left(\{L_0,\rho\} + \sum_{j\neq 0} L_j \rho L_j^\dagger\right)$$
$$= \mathrm{Tr}(\{L_0,\rho\}) + \sum_{j\neq 0} \mathrm{Tr}(L_j \rho L_j^\dagger)$$
$$= \mathrm{Tr}(\{L_0,\rho\}) + \frac{1}{2}\sum_{j\neq 0} \mathrm{Tr}(L_j^\dagger L_j \rho + \rho L_j^\dagger L_j)$$
$$= \mathrm{Tr}(\{L_0,\rho\}) + \mathrm{Tr}\left(\frac{1}{2}\sum_j \{L_j^\dagger L_j, \rho\}\right) = 0$$

Therefore $L_0 = -\frac{1}{2}\sum_j L_j^\dagger L_j$.

2. $L_1 = \gamma |0\rangle\langle 1|$ and $L_2 = \gamma |1\rangle\langle 0|$

a) $\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] + L_1\rho L_1^\dagger + L_2\rho L_2^\dagger - \frac{1}{2}\{L_1^\dagger L_1, \rho\} - \frac{1}{2}\{L_2^\dagger L_2, \rho\}$

$= L_1\rho L_1^\dagger + L_2\rho L_2^\dagger - \frac{1}{2}\{L_1^\dagger L_1 + L_2^\dagger L_2, \rho\}$

$L_1^\dagger L_1 = \gamma^2 |1\rangle\langle 1|$ and $L_2^\dagger L_2 = \gamma^2 |0\rangle\langle 0|$

b) $\rho = \frac{\mathbb{I} + Z}{2} = |0\rangle\langle 0|$, $\frac{d\mathbb{I}}{dt} = 0$

$\frac{d\rho}{dt} = -\gamma^2\left(\frac{\mathbb{I}+Z}{2}\right) + \gamma^2 L_1\left(\frac{\mathbb{I}+Z}{2}\right)L_1^\dagger + \gamma^2 L_2\left(\frac{\mathbb{I}+Z}{2}\right)L_2^\dagger$

$= -\gamma^2\left(\frac{\mathbb{I}+Z}{2}\right) + \frac{L_1L_1^\dagger + L_2L_2^\dagger + L_1ZL_1^\dagger + L_2ZL_2^\dagger}{2}$

$\frac{1}{2}\frac{dZ}{dt} = -\gamma^2\frac{\mathbb{I}}{2} - \gamma^2\frac{Z}{2} + \gamma^2\frac{\mathbb{I}}{2} + -\gamma^2\frac{Z}{2} = -\gamma^2 Z$

$\frac{dZ}{dt} = -2\gamma^2 Z$ so $Z(t) = e^{-2\gamma^2 t} Z$.

$\rho(t) = \frac{\mathbb{I} + e^{-2i\gamma^2 t} Z}{2} = \frac{1}{2}\begin{pmatrix} 1+e^{-2\gamma^2 t} & 0 \\ 0 & 1-e^{-2\gamma^2 t} \end{pmatrix}$

Equilibrium: $t \to 0$ $\rho \to \frac{\mathbb{I}}{2}$.

$\rho = \frac{\mathbb{I}+X}{2} = |+\rangle\langle +|$

$\frac{d\rho}{dt} = -\gamma^2\left(\frac{\mathbb{I}+X}{2}\right) + \frac{L_1L_1^\dagger + L_2L_2^\dagger + L_1XL_1^\dagger + L_2XL_2^\dagger}{2}$

$L_1XL_1^\dagger = \gamma^2 |0\rangle\langle 1|\left(|1\rangle\langle 0| + |0\rangle\langle 1|\right)|1\rangle\langle 0| = 0$

$L_2XL_2^\dagger = 0$

$$\frac{1}{2}\frac{dX}{dt} = -\gamma^2 \frac{\mathbb{I}}{2} - \gamma^2 \frac{X}{2} + \gamma^2 \frac{\mathbb{I}}{2} + 0 = -\gamma^2 \frac{X}{2}$$

So $X(t) = e^{-\gamma^2 t} X$

$$\rho(t) = \frac{1}{2}\begin{pmatrix} 1 & e^{-\gamma^2 t} \\ e^{-\gamma^2 t} & 1 \end{pmatrix} \qquad t \to \infty \quad \rho(t) = \frac{\mathbb{I}}{2}.$$

c) $|0\rangle\langle 0|$: $\quad S(t) = -\sum_k p_k(t) \ln p_k(t)$

$$= -\frac{1+e^{-2\gamma^2 t}}{2} \ln\left(\frac{1+e^{-2\gamma^2 t}}{2}\right) - \frac{1-e^{-2\gamma^2 t}}{2} \ln\left(\frac{1-e^{-2\gamma^2 t}}{2}\right)$$

$|+\rangle\langle +|$: find probabilities (eigenvalues) of $\frac{1}{2}\begin{pmatrix} 1 & e^{-\gamma^2 t} \\ e^{-\gamma^2 t} & 1 \end{pmatrix}$

$$\begin{vmatrix} \frac{1-\lambda}{2} & \frac{e^{-\gamma^2 t}}{2} \\ \frac{e^{-\gamma^2 t}}{2} & \frac{1-\lambda}{2} \end{vmatrix} = \frac{(1-\lambda)^2}{4} - \frac{e^{-2\gamma^2 t}}{4} = 0$$

$$\lambda^2 - 2\lambda + 1 - e^{-2\gamma^2 t} = 0$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4(1-e^{-2\gamma^2 t})}}{2} = 1 \pm \sqrt{e^{-2\gamma^2 t}} = 1 \pm e^{-\gamma^2 t}$$

Normalised probabilities: $p_1 = \frac{1+e^{-2\gamma^2 t}}{2}$, $p_2 = \frac{1-e^{-2\gamma^2 t}}{2}$.

Exactly the same entropy as for $|0\rangle\langle 0|$.

3. $\rho^{T} = \frac{1}{2}\left(|00\rangle\langle 00| + |10\rangle\langle 01| + |01\rangle\langle 10| + |11\rangle\langle 11|\right)$

Translate this into a matrix on the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$\rho^{T} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Find the eigenvalues: $\det(\rho^{T} - \lambda \mathbb{I}) = 0$.

$$\det \begin{vmatrix} \frac{1}{2}-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\lambda & 0 \\ 0 & 0 & 0 & \frac{1}{2}-\lambda \end{vmatrix} = \left(\tfrac{1}{2}-\lambda\right)^{2}\left(\lambda^{2} - \tfrac{1}{4}\right) = 0$$

$$\left(\lambda - \tfrac{1}{2}\right)^{3}\left(\lambda + \tfrac{1}{2}\right) = 0 \rightarrow \lambda = +\tfrac{1}{2} \ (3\times) \text{ and } \underline{\underline{\lambda = -\tfrac{1}{2}}}$$

negative.