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<u>Entanglement - Enhanced Quantum Measurements</u> QIP IRC Summerschool 6-10 June '05

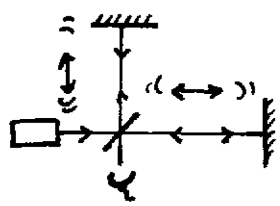
<u>Quantum Information Theory</u>: physical limits to information processing.

discrete variable : perfect read out (such as a qubit)
continuous variable: intrinsic error → <u>what is the error?</u>

Why are continuous variables interesting?

$$|\psi(0)\rangle \longrightarrow \boxed{\varphi} \longrightarrow |\psi(t)\rangle$$

Slab of material in an interferometer or <u>gravitational waves</u>



So we need to know both φ and $\Delta\varphi$. What's more, this must be given in terms of <u>all</u> the resources that are needed (including time).

From the cartoon above, you can see this problem has something to do with the <u>evolution of states.</u>

Also, the only perfectly distinguishable states are <u>orthogonal states</u>, so let's look at the time it takes for a state to evolve to an orthogonal state.

The Mandelstam - Tamm inequality (1945).

We know from QM that $\Delta A \, \Delta B \geq \frac{1}{2} \left| \langle [A, B] \rangle \right|$.

Assume that A is time independent, such that

$i\hbar \dfrac{d\langle A \rangle}{dt} = \langle [A, B] \rangle$, then we have

$$\Delta H \, \Delta A \geq \frac{\hbar}{2} \left| \frac{d\langle A \rangle}{dt} \right| .$$

We want to know how long it takes to get

$$|\psi(0)\rangle \rightarrow |\psi(t)\rangle = e^{iHt/\hbar} |\psi(0)\rangle$$

Choose $A = |\psi(0)\rangle\langle\psi(0)|$. We then have

$$\langle A \rangle = \langle \psi(t) | A | \psi(t) \rangle = |\langle \psi(0) | \psi(t) \rangle|^2 \equiv \cos^2 \phi(t)$$

$$(\Delta A)^2 = \langle \psi(t) | A^2 | \psi(t) \rangle - \langle \psi(t) | A | \psi(t) \rangle^2$$

$$= \langle A \rangle \left(1 - \langle A \rangle \right)$$

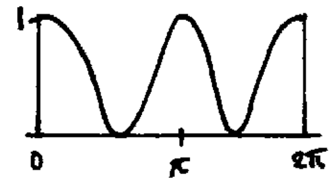
So $\Delta A = \sqrt{\cos^2 \phi \, (1 - \cos^2 \phi)} = \cos\phi \sin\phi$. The inequality above then becomes

$$\Delta H \cos\phi \sin\phi \geq \frac{\hbar}{2} \left| \frac{d \cos^2 \phi(t)}{dt} \right| = \hbar \cos\phi \sin\phi \left| \frac{d\phi}{dt} \right|$$

After integration: $|\phi(t)| \leq \dfrac{\Delta H \, t}{\hbar}$

Feed this result back into $|\langle\psi(0)|\psi(t)\rangle|^2 = \cos^2\phi$

$$\phi \leq \frac{\Delta H t}{\hbar} \quad \Rightarrow \quad \cos^2\phi \geq \cos^2\left(\frac{\Delta H t}{\hbar}\right)$$



$$|\langle\psi(0)|\psi(t)\rangle|^2 \geq \cos^2\left(\frac{\Delta H t}{\hbar}\right)$$

Orthogonality then gives $0 \geq \cos\left(\frac{\Delta H t}{\hbar}\right)$, or

$$\frac{\pi}{2} \leq \frac{\Delta H t}{\hbar}$$

This leads to $\boxed{t \geq \frac{\pi}{2}\frac{\hbar}{\Delta H},}$ the Mandelstam–Tamm equation.

Sometimes, this bound is not applicable, for example when $\Delta H \to \infty, 0$.

<u>The Margolus – Levitin inequality (1998)</u>

Without loss of generality, we can define $|\psi(0)\rangle$ as a superposition of energy eigenstates with $E_0 = 0$:

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n |E_n\rangle .$$

The time evolution is then given by

$$|\psi(t)\rangle = \sum_{n=0}^{\infty} c_n e^{-iE_n t/\hbar} |E_n\rangle .$$

Define $S(t) = \langle \psi(0) | \psi(t) \rangle = \sum_{n=0}^{\infty} |c_n|^2 e^{-iE_n t/\hbar}$.

We then want to find the smallest value of t such that $S = 0$.

$$\begin{aligned}
\mathrm{Re}(S) &= \sum_{n=0}^{\infty} |c_n|^2 \cos\left(\frac{E_n t}{\hbar}\right) \\
&\geq \sum_{n=0}^{\infty} |c_n|^2 \left(1 - \frac{2}{\pi}\left(\frac{E_n t}{\hbar} + \sin\left(\frac{E_n t}{\hbar}\right)\right)\right) \\
&= 1 - \frac{2}{\pi}\frac{\langle E \rangle t}{\hbar} + \frac{2}{\pi}\mathrm{Im}(S).
\end{aligned}$$

In the second line, we used

$$\cos x \geq 1 - \frac{2}{\pi}(x + \sin x) \quad \text{for } x \geq 0$$

Observe: $S = 0 \rightarrow \mathrm{Re}(S) = \mathrm{Im}(S) = 0$:

$$0 \geq 1 - \frac{2}{\pi}\frac{\langle E \rangle t}{\hbar} \quad \text{or} \quad \boxed{t \geq \frac{\pi}{2}\frac{\hbar}{\langle E \rangle}}.$$

We call this the Margolus-Levitin inequality.

<u>Conclusion</u>: we need to know both $\langle E \rangle$ and ΔH of a state to find its dynamical speed of evolution.

For metrology, we need to choose our states such that both $\langle E \rangle$ and ΔH are favourable.

Why are these inequalities relevant?

Well, for example, if we look at a single optical mode, we have $\Delta\phi = \omega t$, and

$$\boxed{\Delta\phi \geq \frac{\pi}{2}\frac{\hbar\omega}{\Delta H} \qquad \text{and} \qquad \Delta\phi \geq \frac{\pi}{2}\frac{\hbar\omega}{\langle E\rangle}}$$

So we have two expressions for the minimal resolvable phase $\Delta\phi$, given an interaction time t and Hamiltonian H.

Let's look at two optical implementations.

<u>coherent state</u>: $\quad |\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \,, \quad H = \hbar\omega(\hat{n} + \tfrac{1}{2})$

$\langle E\rangle = \langle\alpha|H|\alpha\rangle \simeq \hbar\omega|\alpha|^2 = \hbar\omega\langle n\rangle$, with $\langle n\rangle$ the mean number of photons.

$$\Delta H = \sqrt{\langle\alpha|H^2|\alpha\rangle - \langle\alpha|H|\alpha\rangle^2}$$

$$\langle\alpha|H^2|\alpha\rangle = \hbar^2\omega^2|\alpha|^2(|\alpha|^2 + 1)$$

$$\Delta H \simeq \hbar\omega\sqrt{\langle n\rangle}$$

$$\left.\begin{aligned} \text{M.T.}: &\quad t_{MT} \geq \frac{\pi}{2}\frac{\hbar}{\Delta H} = \frac{\pi}{2}\frac{1}{\omega\sqrt{\langle n\rangle}} \\ \text{M.L.}: &\quad t_{ML} \geq \frac{\pi}{2}\frac{\hbar}{\langle E\rangle} = \frac{\pi}{2}\frac{1}{\omega\langle n\rangle} \end{aligned}\right\} \quad t_{MT} \geq t_{ML}$$

Taking $\Delta\varphi \geq \pi$ [illegible] this is called the Standard Quantum Limit

<u>Noon states</u> $|N,0\rangle + |0,N\rangle$, an energy eigenstate.

$\langle E \rangle \cong \hbar\omega N$ and $\Delta H = 0$ (I suppressed the vacuum)

M.T. : $t_{MT} \geq \frac{\pi}{2} \frac{\hbar}{\Delta H}$ ↯ divide by zero

M.L. : $t_{ML} \geq \frac{\pi}{2} \frac{1}{\omega N}$ ← This is called the Heisenberg Limit.

So it is <u>not</u> true that the MT is the SQL and ML is the HL. Rather, the MT and ML determine whether a given setup reaches the SQL, or the HL.

Entanglement assisted phase estimation

Let's first look at the 'classical' case. We prepare N qubits in the state

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

The evolution of the qubit is given by

$|0\rangle \to |0\rangle$ and $|1\rangle \to e^{i\varphi}|1\rangle$, yielding

$$|+\rangle \to |\varphi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right).$$

We have to find a suitable observable to estimate φ:

$$\sigma_x = |0\rangle\langle 1| + |1\rangle\langle 0|$$

with expectation value $\langle\sigma_x\rangle = \langle\varphi|\sigma_x|\varphi\rangle = \cos\varphi$

<u>Repeat this with N qubits</u>, and we find $\langle\sigma_x^{(N)}\rangle^2 = N\cos^2\varphi$

Using $\sigma_x^2 = \mathbb{1}$, we get $\langle\sigma_x^{2(N)}\rangle = N$

$$(\Delta\sigma_x^{(N)})^2 = N - N\cos^2\varphi = N\sin^2\varphi$$

From estimation theory, we have $\Delta A = \left|\frac{d\langle B\rangle}{dA}\right|^{-1}\Delta B$:

$$\Delta\varphi = \left|\frac{d\langle\sigma_x^{(N)}\rangle}{d\varphi}\right|^{-1}\Delta\sigma_x^{(N)} = \frac{\sqrt{N}\sin\varphi}{N\sin\varphi} = \frac{1}{\sqrt{N}}$$

Let's see how <u>entanglement</u> can help:

$$|+_N\rangle = \frac{1}{\sqrt{2}}\left(|0,\dots,0\rangle + |1,\dots,1\rangle\right)$$

with the same evolution. This yields

$$|\varphi_N\rangle = \frac{1}{\sqrt{2}}\left(|0,\dots,0\rangle + e^{iN\varphi}|1,\dots,1\rangle\right)$$

Redefine $|0,\dots,0\rangle \equiv |\vec{0}\rangle$ and $|1,\dots,1\rangle \equiv |\vec{1}\rangle$

$$|\varphi_N\rangle = \frac{1}{\sqrt{2}}\left(|\vec{0}\rangle + e^{iN\varphi}|\vec{1}\rangle\right)$$

This is mathematically equivalent to a single (nonlocal!) system with a relative phase shift $N\varphi$.

Choose $\Sigma_N \equiv |\vec{0}\rangle\langle\vec{1}| + |\vec{1}\rangle\langle\vec{0}|$, then

$$\langle\varphi_N|\Sigma_N|\varphi_N\rangle = \cos N\varphi \quad \text{and} \quad \langle\varphi_N|\Sigma_N^2|\varphi_N\rangle = 1$$

$$\Delta\Sigma_N = \sqrt{1-\cos^2 N\varphi} = \sin N\varphi$$

$$\Delta\varphi = \left|\frac{d\langle\Sigma_N\rangle}{d\varphi}\right|^{-1}\Delta\Sigma_N = \frac{\sin N\varphi}{N\sin N\varphi} = \frac{1}{N}.$$

This is the <u>Heisenberg limit</u>; Compare to the Standard Quantum Limit:

$$\Delta\varphi = \frac{1}{\sqrt{N}}.$$

Quantum Lithography

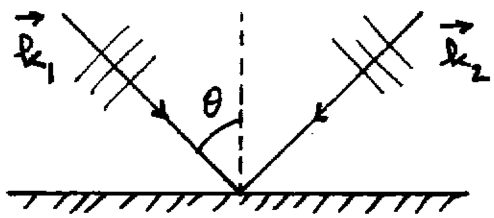
We saw that the state $|\vec{0}\rangle + e^{iN\varphi}|\vec{1}\rangle$ gives the expectation value

$$\langle \varphi_N | \Sigma_N | \varphi_N \rangle = \cos N\varphi .$$

The factor N in the argument means that there are N maxima when φ runs through its domain $[0, 2\pi)$.

Question: Can we exploit this?

Rayleigh limit



Interference pattern: $I(\vec{r}) \propto \left| e^{i\vec{k}_1\cdot\vec{r}} + e^{i\vec{k}_2\cdot\vec{r}} \right|^2$

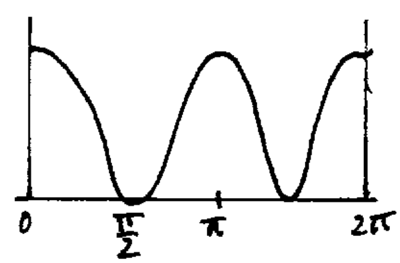
Let $|\vec{k}_1| = |\vec{k}_2| = k$ and $k = \frac{2\pi}{\lambda}$:

$$I(\vec{r}) = 4\cos^2\left[\tfrac{1}{2}(\vec{k}_1 - \vec{k}_2)\cdot\vec{r}\right] \propto \cos^2(k x \sin\theta)$$

The minimum feature size that is resolvable

$$k\,\Delta x \sin\theta = \frac{\pi}{2}, \quad \text{or}$$

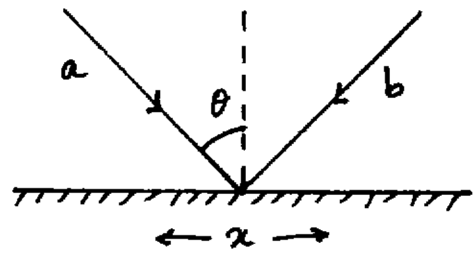
$$\Delta x = \frac{\pi}{2k\sin\theta} = \frac{\lambda}{4\sin\theta}$$



In the grazing limit $\theta = \frac{\pi}{2}$, the <u>Rayleigh limit</u> is $\Delta x = \frac{\lambda}{4}$.

Instead of plane waves, now use <u>entangled light</u>:

$$|\psi_N\rangle_{ab} = \frac{1}{\sqrt{2}}\left(|N,0\rangle_{ab} + e^{iN\varphi}|0,N\rangle_{ab}\right)$$



Note the difference between N qubits and N field excitations. However, the mathematics is the same.

Define $\varphi = kx$ and $k = \frac{2\pi}{\lambda}$ $\left(\theta = \frac{\pi}{2}\right)$.

The (classical) field intensity on the substrate is given by

$$\Delta = \langle\psi|E^- E^+|\psi\rangle \propto \langle\psi|\hat{e}^\dagger\hat{e}|\psi\rangle, \text{ with } \hat{e} = \frac{\hat{a}+\hat{b}}{\sqrt{2}}.$$

Using the entangled state $|\psi_N\rangle$, we have to look at the <u>higher-order moments</u> of the field:

$$\Delta_N = \langle\psi_N|(E^-)^N(E^+)^N|\psi_N\rangle \equiv \langle\psi_N|\delta_N|\psi_N\rangle,$$

$$\text{with} \quad \delta_N = \frac{(\hat{e}^\dagger)^N\,\hat{e}^N}{N!}$$

The factorial $N!$ takes into account the overcounting when N photons are used to bridge the energy gap $N\hbar\omega$ in the substrate.

We find that $\Delta_N \propto (1+\cos N\varphi) = (1+\cos Nkx)$

The <u>quantum Rayleigh limit</u> is then $\Delta x = \frac{\pi}{Nk} = \frac{\lambda}{2N}$.

Entanglement-Enhanced Quantum Measurements
Problem Sheet

QIP IRC Summer school

5–10 June, 2005

1. Given the beam-splitter transformation

$$\hat{a}_1 = \frac{\hat{a}_1' + i\hat{a}_2'}{\sqrt{2}} \quad \text{and} \quad \hat{a}_2 = \frac{i\hat{a}_1' + \hat{a}_2'}{\sqrt{2}} \, ,$$

show that the mode transformations of the Mach-Zehnder interferometer in Fig. 1 are given by:

$$\begin{aligned} \hat{b}_1 &= \frac{1}{2}(1 + e^{-i\phi})\hat{a}_1 + \frac{i}{2}(-1 + e^{-i\phi})\hat{a}_2 \, , \\ \hat{b}_2 &= \frac{i}{2}(1 - e^{-i\phi})\hat{a}_1 + \frac{1}{2}(1 + e^{-i\phi})\hat{a}_2 \, . \end{aligned}$$

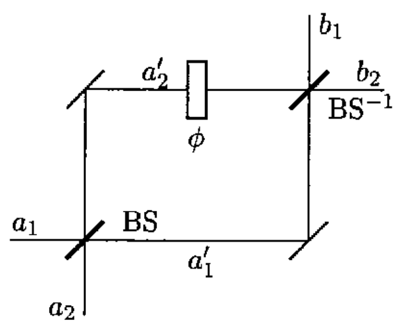


Figure 1: The Mach-Zehnder interferometer.

(a) Calculate the error in the phase $\Delta\phi$ for the single-photon input state $|1, 0\rangle_{a_1 a_2}$. Assume that we measure the photon number difference in the output modes;

(b) similarly for the input state $|N, 0\rangle_{a_1 a_2}$;

(c) and for the input state $(|N, N-1\rangle_{a_1 a_2} + |N-1, N\rangle_{a_1 a_2})/\sqrt{2}$. Is this optimal?

2. Show that $\Delta_N \propto (1 + \cos N\phi)$ in quantum lithography.